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Quaternionic Formulation of Supersymmetric Quantum Mechanics

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Abstract Quaternionic formulation of supersymmetric quantum mechanics has been developed consistently in terms of Hamiltonians, super partner Hamiltonians, and supercharges for free particle and interacting field in one and three dimensions. Supercharges, super partner Hamiltonians and energy eigenvalues are discussed and it has been shown that the results are consistent with the results of quantum mechanics.

Keywords Quaternion · Supersymmetry · Quantum Mechanics

1 Introduction

Quaternionic quantum mechanics has been extensively studied by Adler [1], while other authors [2, 3] revealed out the noble features of quaternionic quantum mechanics. But the subject has not been considered widely since there are various problems with non-commutative nature of quaternion multiplication besides the advance algebraic structure. On the other hand, supersymmetric quantum mechanics is an application of SUSY super algebra to quantum mechanics as approved by quantum field theory. So one-dimensional SUSY has been studied by various authors [4, 5], and the efforts have been made by various authors [5–11] to generalize it to higher dimensions. One of the attempts was also made by Das et al. [2] to consider the higher dimensional SUSY quantum mechanics. While the Cooper et al. [5] reviewed the theoretical formulation of quantum mechanics and discussed many problems therein. Supersymmetric quantum mechanics involves pairs of Hamiltonians, which share a particular mathematical relationship, which are called partner Hamiltonians and the potential energy terms occur in Hamiltonians are then described as partner potentials. Accordingly, for every eigenstate, of one Hamiltonian in partner Hamiltonian, has a corresponding

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eigenstate with the same energy (except possible for zero energy eigenstates). Each boson would have a fermionic partner of eigen energy but in relativistic world energy and mass are interchangeable. So one can say that partner particles have equal masses. SUSY concepts have provided useful extension to WKB approximation [6]. Supersymmetric methods in quaternionic quantum mechanics are discussed by Adler [1] and Davies [12] to study supersymmetric quaternionic quantum mechanics.

Keeping in view the application of SUSY and quaternion quantum mechanics, we have made an attempt in this paper to develop quaternionic quantum mechanics from the basics of free particle quaternion differential operator. Free particle super partner Hamiltonian, supercharges and total Hamiltonian are accordingly calculated. Introducing the interaction through quaternion super potential, interacting super-partner Hamiltonians, supercharges and total Hamiltonian are discussed consistently and satisfies the properties of supersymmetric algebra. Because of non-commutative nature of quaternions, we have made an attempt to solve the problem by restricting the propagation along X-axis only and interacting operators in one dimension are derived. Correspondingly, the supercharges, super partner Hamiltonians and total Hamiltonian are again discussed to satisfy the SUSY algebra. It has been shown that the condition for good supersymmetry is that supercharges must annihilate the vacuum state. Using this condition, we have obtained ground state quaternionic wave function. It has been shown that the quaternionic super potential obtained in this manner resembles with the result obtained earlier by Davies [12]. With the help of these operators Schrodinger wave equation is obtained for Hamiltonian. Super partner Hamiltonians are factorized in terms of creation and annihilation operators and in that case our results resemble with Sukumar [10]. It has been shown that energy eigenvalues of super partner Hamiltonians are positive definite. The ground state wave function has also been obtained in terms of quaternion potential and super partner Hamiltonians are derived consistently. It has also been shown that the second order super potential describes anti-commutation relations while the first order super potential gives rise to commutation relations of creation and annihilation operators. As such the first and second order super potential describes respectively the system of bosons and fermions. it has been calculated that the energy eigenvalue of super partner Hamiltonian is no vanishing but equals to the energy of first excited state. It has also been shown that the energy of in first excited equals to energy of in second excited state. We have also shown that the energy spectrum is related as energy eigenstates are equally spaced. Our results are same as those obtained earlier by Sukumar [10] and Rajput [13] and we may conclude that quaternionic supersymmetric quantum mechanics is consistent with supersymmetric quantum mechanics.

2 Definition

A quaternion ϕ is expressed as

$$\phi = e_0\phi_0 + e_1\phi_1 + e_2\phi_2 + e_3\phi_3 \tag{1}$$

where $\phi_0, \phi_1, \phi_2, \phi_3$ are the real quartets of a quaternion and e_0, e_1, e_2, e_3 are called quaternion units and satisfies the following relations,

$$\begin{aligned} e_0^2 &= e_0 = 1, \\ e_0 e_i &= e_i e_0 = e_i \quad (i = 1, 2, 3), \\ e_i e_j &= -\delta_{ij} + \varepsilon_{ijk} e_k \quad (i, j, k = 1, 2, 3) \end{aligned} \tag{2}$$

where δ_{ij} is the Kronecker delta and ε_{ijk} is the three index Levi-Civita symbols with their usual definitions. The quaternion conjugate $\bar{\phi}$ is then defined as

$$\bar{\phi} = e_0\phi_0 - e_1\phi_1 - e_2\phi_2 - e_3\phi_3 \tag{3}$$

Here ϕ_0 is real part of the quaternion defined as

$$\phi_0 = Re\,\phi = \frac{1}{2}(\bar{\phi} + \phi) \tag{4}$$

If $Re\,\phi = \phi_0 = 0$, then $\phi = -\bar{\phi}$ and imaginary ϕ is called pure quaternion and is written as

$$\operatorname{Im}\phi = e_1\phi_1 + e_2\phi_2 + e_3\phi_3 \tag{5}$$

The norm of a quaternion is expressed as

$$N(\phi) = \bar{\phi}\phi = \phi\bar{\phi} = \phi_0^2 + \phi_1^2 + \phi_2^2 + \phi_3^2 \geq 0 \tag{6}$$

and the inverse of a quaternion is described as

$$\phi^{-1} = \frac{\bar{\phi}}{|\phi|} \tag{7}$$

While the quaternion conjugation satisfies the following property

$$\overline{(\phi_1\phi_2)} = \bar{\phi}_2\bar{\phi}_1 \tag{8}$$

The norm of the quaternion (6) is positive definite and enjoys the composition law

$$N(\phi_1\phi_2) = N(\phi_1)N(\phi_2) \tag{9}$$

Quaternion (1) is also written as $\phi = (\phi_0, \overrightarrow{\phi})$ where $\overrightarrow{\phi} = e_1\phi_1 + e_2\phi_2 + e_3\phi_3$ is its vector part and ϕ_0 is its scalar part. The sum and product of two quaternions are

$$\begin{aligned}(\alpha_0, \overrightarrow{\alpha}) + (\beta_0, \overrightarrow{\beta}) &= (\alpha_0 + \beta_0, \overrightarrow{\alpha} + \overrightarrow{\beta}),\\ (\alpha_0, \overrightarrow{\alpha})\,(\beta_0, \overrightarrow{\beta}) &= (\alpha_0\beta_0 - \overrightarrow{\alpha}.\overrightarrow{\beta}\,,\, \alpha_0\overrightarrow{\beta} + \beta_0\overrightarrow{\alpha} + \overrightarrow{\alpha} \times \overrightarrow{\beta})\end{aligned} \tag{10}$$

Quaternion elements are non-Abelian in nature and thus represent a division ring.

3 Quaternion SUSY for Free Particle

Let us define four differential operator as quaternion in the following manner (on using natural units $c = \hbar = 1$ and $i = \sqrt{-1}$ through out the text);

$$\boxdot = e_1\partial_1 + e_2\partial_2 + e_3\partial_3 + \partial_4 = -i\,\frac{\partial}{\partial t} + e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} + e_3\frac{\partial}{\partial x_3} \tag{11}$$

The quaternion conjugate of this equation is described as

$$\overline{\boxdot} = -e_1\partial_1 - e_2\partial_2 - e_3\partial_3 + \partial_4 = -i\,\frac{\partial}{\partial t} - e_1\frac{\partial}{\partial x_1} - e_2\frac{\partial}{\partial x_2} - e_3\frac{\partial}{\partial x_3} \tag{12}$$

Using the quaternion multiplication rule (2) we may write the norm of the quaternion differential operator given by (11–12) as

$$N(\boxdot) = \boxdot\overline{\boxdot} = \overline{\boxdot}\boxdot = \partial_1^2 + \partial_2^2 + \partial_3^2 + \partial_4^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial t^2} \tag{13}$$

Equation (13) can also be related with the D'Alembertian operator in the fallowing manner i.e.

$$\square = \boxdot\overline{\boxdot} = \overline{\boxdot}\boxdot = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial t^2} = -\frac{\partial^2}{\partial t^2} + \nabla^2 \tag{14}$$

Let us consider the quaternion differential operator (in three space dimensions) and describe it as the free particle operator i.e.

$$\widehat{A}_{free} = \triangle = e_1\partial_1 + e_2\partial_2 + e_3\partial_3 = \sum_{j=1}^{3} e_j\,\nabla_j \quad \left(\nabla_j = \frac{\partial}{\partial x_j}\right) \tag{15}$$

The conjugate of (15) is then be written as

$$\widehat{A}^{\dagger}_{free} = \triangle^{\dagger} = -e_1\partial_1 - e_2\partial_2 - +e_3\partial_3 = \sum_{j=1}^{3} e_j^{\dagger}\nabla_j^{\dagger} \tag{16}$$

where † corresponds to quaternionic conjugation. So that super partner of a free particle Hamiltonian can be formed as

$$\begin{aligned} \hat{H}_1 = \hat{H}_- = \widehat{A}^{\dagger}_{free}\widehat{A}_{free} = \sum_{j=1}^{3} e_j^{\dagger}\,\nabla_j^{\dagger}\cdot\sum_{j=1}^{3} e_j\nabla_j, \\ \hat{H}_2 = \hat{H}_+ = \widehat{A}_{free}\widehat{A}^{\dagger}_{free} = \sum_{j=1}^{3} e_j\,\nabla_j\cdot\sum_{j=1}^{3} e_j^{\dagger}\nabla_j^{\dagger} \end{aligned} \tag{17}$$

Accordingly, the supercharges are described as

$$\begin{aligned} \hat{Q} = \begin{bmatrix} 0 & \widehat{A}_{free} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sum_{j=1}^{3} e_j\nabla_j \\ 0 & 0 \end{bmatrix} \\ \hat{Q}^{\dagger} = \begin{bmatrix} 0 & 0 \\ \widehat{A}^{\dagger}_{free} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \sum_{j=1}^{3} e_j\nabla_j & 0 \end{bmatrix} \end{aligned} \tag{18}$$

So the free particle Hamiltonian in 3-dimensions is described as

$$\hat{H}_{free} = \widehat{H} = \left\{ \begin{matrix} \sum_{j=1}^{3} e_j\,\nabla_j\cdot\sum_{j=1}^{3} e_j^{\dagger}\nabla_j^{\dagger} & 0 \\ 0 & \sum_{j=1}^{3} e_j^{\dagger}\,\nabla_j^{\dagger}\cdot\sum_{j=1}^{3} e_j\nabla_j \end{matrix} \right\} \tag{19}$$

Here the supercharges (18) and Hamiltonian (19) satisfy the super symmetric algebra given by

$$\begin{aligned} [\widehat{H}, \hat{Q}] = [\widehat{H}, \hat{Q}^{\dagger}] = 0, \\ \{\hat{Q}, \hat{Q}\} = \{\hat{Q}^{\dagger}, \hat{Q}^{\dagger}\} = 0 \end{aligned} \tag{20}$$

Here (20) results in degeneracy of energy and hence, the supercharges $\hat{Q}$ and $\hat{Q}^\dagger$ and generate supersymmetric transformations and change accordingly a bosonic state to a fermionic state or vice versa. Relation $\widehat{H} = \{\hat{Q}, \hat{Q}^\dagger\}$ shows that Hamiltonian can have only positive or zero eigen values i.e.

$$\begin{aligned} \langle \psi \mid \widehat{H} \mid \psi \rangle &= \langle \psi \mid \hat{Q}\hat{Q}^\dagger \mid \psi \rangle + \langle \psi \mid \hat{Q}\hat{Q}^\dagger \mid \psi \rangle \\ &= \left|\hat{Q} \mid \psi \right\rangle^2 + \left|\hat{Q}^\dagger \mid \psi \right\rangle^2 \geq 0 \end{aligned} \tag{21}$$

4 Quaternion SUSY for Interacting Field

Let us describe the interaction through the introduction of quaternion super potential defined as

$$U = e_1 U_1 + e_2 U_2 + e_3 U_3 \tag{22}$$

where $\overrightarrow{U} = (U_1, U_2, U_3)$ is the three dimensional super potential. Then, the operators for this case of interaction become

$$\begin{aligned} \widehat{A} &= \boxdot + U = \sum_{j=1}^{3} e_j (\nabla_j + U_j), \\ \widehat{A}^\dagger &= \boxdot^\dagger + U^\dagger = \sum_{j=1}^{3} e_j^\dagger (\nabla_j^\dagger + U_j^\dagger) \end{aligned} \tag{23}$$

So that we may define the supercharges for interacting field as follows,

$$\begin{aligned} \hat{Q} &= \begin{bmatrix} 0 & \widehat{A} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sum_{j=1}^{3} e_j (\nabla_j + U_j) \\ 0 & 0 \end{bmatrix} \\ \hat{Q}^\dagger &= \begin{bmatrix} 0 & 0 \\ \widehat{A}^\dagger & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \sum_{j=1}^{3} e_j^\dagger (\nabla_j^\dagger + U_j^\dagger) & 0 \end{bmatrix} \end{aligned} \tag{24}$$

while the super partner Hamiltonians are defined in the following manner

$$\begin{aligned} \hat{H}_1 = \hat{H}_- = \widehat{A}^\dagger \widehat{A} &= \sum_{j=1}^{3} e_j^\dagger (\nabla_j^\dagger + U_j^\dagger) \cdot \sum_{j=1}^{3} e_j (\nabla_j + U_j) \\ \hat{H}_2 = \hat{H}_+ = \widehat{A}\widehat{A}^\dagger &= \sum_{j=1}^{3} e_j (\nabla_j + U_j) \cdot \sum_{j=1}^{3} e_j^\dagger (\nabla_j^\dagger + U_j^\dagger). \end{aligned} \tag{25}$$

So that Hamiltonian from equation is described as

$$\widehat{H} = \left\{\hat{Q}, \hat{Q}^\dagger\right\} = \begin{bmatrix} \widehat{A}\widehat{A}^\dagger & 0 \\ 0 & \widehat{A}^\dagger \widehat{A} \end{bmatrix} = \begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix} \tag{26}$$

or

$$\widehat{H} = \begin{Bmatrix} \sum_{j=1}^{3} e_j (\nabla_j + U_j) \cdot \sum_{j=1}^{3} e_j^\dagger (\nabla_j^\dagger + U_j^\dagger) & 0 \\ 0 & \sum_{j=1}^{3} e_j^\dagger (\nabla_j^\dagger + U_j^\dagger) \cdot \sum_{j=1}^{3} e_j (\nabla_j + U_j) \end{Bmatrix} \tag{27}$$

As such, we can verify the algebra of Supersymmetry (SUSY) i.e.

$$\begin{aligned} &[\widehat{Q},\widehat{H}]=[\widehat{Q},\widehat{H}^\dagger]=0,\\ &\{\widehat{Q},\widehat{Q}\}=\{\widehat{Q}^\dagger,\widehat{Q}^\dagger\}=0,\\ &\{\widehat{Q},\widehat{Q}^\dagger\}=\widehat{H} \end{aligned} \tag{28}$$

which is same as that of (20). As such, the SUSY is satisfied for the case of interacting field for which the quaternionic formulation of supercharges and Hamiltonian are described by (24) and (27).

Let us restrict the propagation along one dimension (say X-axis only) and letting $Y=Z=0$, for simplification, and choosing quaternionic unit e_2 along x-axis. Then the annihilation and creation operators respectively given by $\widehat{A}^\dagger$ and $\widehat{A}$ are derived as

$$\begin{aligned} \widehat{A}&=e_2\frac{d}{dx}+\widehat{U}(x),\\ \widehat{A}^\dagger&=e_2\frac{d}{dx}-\widehat{U}(x) \end{aligned} \tag{29}$$

So that supercharges are obtained as

$$\begin{aligned} \hat{Q}&=\begin{bmatrix}0 & \widehat{A}\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0 & e_2\frac{d}{dx}+\widehat{U}(x)\\ 0 & 0\end{bmatrix},\\ \hat{Q}^\dagger&=\begin{bmatrix}0 & 0\\ \widehat{A}^\dagger & 0\end{bmatrix}=\begin{bmatrix}0 & 0\\ e_2\frac{d}{dx}-\widehat{U}(x) & 0\end{bmatrix} \end{aligned} \tag{30}$$

and accordingly we may write the super partner Hamiltonians as

$$\begin{aligned} \hat{H}_1&=\hat{H}_-=\widehat{A}^\dagger\widehat{A}=-\frac{d^2}{dx^2}+e_2\widehat{U}'(x)-\widehat{U}^2(x),\\ \hat{H}_2&=\hat{H}_-=\widehat{A}\widehat{A}^\dagger=-\frac{d^2}{dx^2}-e_2\widehat{U}'(x)-\widehat{U}^2(x) \end{aligned} \tag{31}$$

Hence the total Hamiltonian in one dimension reduces to the following expressions

$$\widehat{H}=\begin{bmatrix}-\frac{d^2}{dx^2}-e_2\widehat{U}'(x)-\widehat{U}^2(x) & 0\\ 0 & -\frac{d^2}{dx^2}+e_2\widehat{U}'(x)-\widehat{U}^2(x)\end{bmatrix} \tag{32}$$

This Hamiltonian Hermitian i.e. $\widehat{H}\ =\widehat{H}^\dagger$ and its eigen values are real contrary to the quaternion quantum mechanics [1]. We may now relate the real and quaternion Hamiltonian [1] in the following manner

$$\widehat{H}=-e_2\widetilde{H}=i\,\widetilde{H} \tag{33}$$

Since e_2 has eigenvalue $\pm i$. Here $\widetilde{H}$ is the quaternionic Hamiltonian defined [1, 12] as

$$\widetilde{H}=\begin{bmatrix}-e_2\frac{d^2}{dx^2}+\widehat{U}'(x)-e_2\widehat{U}^2(x) & 0\\ 0 & -e_2\frac{d^2}{dx^2}-\widehat{U}'(x)-e_2\widehat{U}^2(x)\end{bmatrix} \tag{34}$$

Equation (31) may then now be written as

$$\begin{aligned} \hat{H}_1 = \hat{H}_- = \widehat{A}^{\dagger}\widehat{A} = -\frac{d^2}{dx^2} + \widehat{V}_-(x) = -\frac{d^2}{dx^2} + \widehat{V}_1(x), \\ \hat{H}_2 = \hat{H}_+ = \widehat{A}\widehat{A}^{\dagger} = -\frac{d^2}{dx^2} + \widehat{V}_+(x) = -\frac{d^2}{dx^2} + \widehat{V}_2(x) \end{aligned} \tag{35}$$

where $\widehat{V}_1(x)$ or $\widehat{V}_-(x)$ and $\widehat{V}_2(x)$ or $\widehat{V}_+(x)$ are known as super partner potentials and are thus related to quaternionic potential U in the following manner,

$$\begin{aligned} \widehat{V}_1(x) = -e_2\widehat{U}'(x) - \widehat{U}^2(x), \\ \widehat{V}_2(x) = e_2\widehat{U}'(x) - \widehat{U}^2(x) \end{aligned} \tag{36}$$

Here also we may establish the condition for good supersymmetry which is known as unbroken supersymmetry and where the supercharges annihilate the vacuum i.e.

$$\widehat{Q}\,|\psi_0\rangle = \widehat{Q}^{\dagger}\,|\psi_0\rangle = 0 \tag{37}$$

where the ground state wave function is defined in terms of two component wave function $|\psi_0\rangle = \left[\begin{smallmatrix}\psi_a(x)\\ \psi_b(x)\end{smallmatrix}\right]$ along with $\psi_a(x)$ and $\psi_b(x)$ are again described in terms of two component wave function of a quaternion in symplectic representation i.e.

$$\begin{aligned} \psi_a(x) = \psi_0 + e_1\psi_1, \\ \psi_b(x) = \psi_2 - e_1\psi_3 \end{aligned} \tag{38}$$

Using (30), (37) and (38), we get

$$\begin{aligned} \begin{bmatrix} 0 & e_2\frac{d}{dx} + \widehat{U}(x) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \psi_a(x) \\ \psi_b(x) \end{bmatrix} = e_2\psi_b'(x) + \widehat{U}(x)\psi_b(x) = 0, \\ \begin{bmatrix} 0 & 0 \\ e_2\frac{d}{dx} - \widehat{U}(x) & 0 \end{bmatrix}\begin{bmatrix} \psi_a(x) \\ \psi_b(x) \end{bmatrix} = e_2\psi_a'(x) + \widehat{U}(x)\psi_a(x) = 0 \end{aligned} \tag{39}$$

which leads to the following sets of equations i.e.

$$\widehat{U}(x) = -\frac{\psi'_{a,b}(x)e_2\psi^{\star}_{a,b}(x)}{\left|\psi_{a,b}(x)\right|^2} = \pm\frac{\psi'(x)e_2\psi^{\star}(x)}{|\psi(x)|^2} \tag{40}$$

Since e_2 has eigenvalues $\pm i$. Replacing e_2 by $\pm i$ our theory gives rise to the results obtained by Davies [12] and accordingly, we may obtain the hierarchy of Hamiltonians or a series of Hamiltonians $\widehat{H}_1, \widehat{H}_2, \widehat{H}_3 \ldots \widehat{H}_n$.

Now, we may write Schrödinger's equation for $\widehat{H}_1(\widehat{H}_-)$ as

$$\widehat{H}_1\psi_0^{(1)}(x) = -\frac{d^2\psi_0^{(1)}(x)}{dx^2} + \widehat{V}_-(x)\psi_0^{(1)}(x) \quad (\hbar = 2m = 1) \tag{41}$$

where

$$\begin{aligned} \widehat{H}_1 = \widehat{H}_- = \widehat{A}_1^{\dagger}\widehat{A}_1 = -\frac{d^2}{dx^2} + \widehat{V}_-(x) = -\frac{d^2}{dx^2} + \widehat{V}_1(x), \\ \widehat{A}_1 = e_2\frac{d}{dx} + \widehat{U}_1(x), \\ \widehat{A}_1^{\dagger} = e_2\frac{d}{dx} - \widehat{U}_1(x) \end{aligned} \tag{42}$$

Similarly, we get Schrödinger's equation for $\widehat{H}_2(\widehat{H}_+)$ as

$$\widehat{H}_2\psi_0^{(1)}(x) = -\frac{d^2\psi_0^{(1)}(x)}{dx^2} + \widehat{V}_+(x)\psi_0^{(1)}(x) \quad (\hbar = 2m = 1)$$
$$\widehat{H}_2 = \widehat{H}_+ = \widehat{A}_1\widehat{A}_1^\dagger = -\frac{d^2}{dx^2} + \widehat{V}_+(x) = -\frac{d^2}{dx^2} + \widehat{V}_2(x) \tag{43}$$

As such, we may obtain the positive energy eigenvalues of both super partner Hamiltonians $\widehat{H}_1(\widehat{H}_-)$ and $\widehat{H}_2(\widehat{H}_+)$ and it is to be proved that operator $\widehat{A}$ converts the eigenstate of $\widehat{H}_1(\widehat{H}_-)$ into the eigenstate of $\widehat{H}_2(\widehat{H}_+)$. Similarly operator $\widehat{A}^\dagger$ converts eigenstate of $\widehat{H}_2(\widehat{H}_+)$ into eigenstate of $\widehat{H}_1(\widehat{H}_-)$. Thus we conclude that $\widehat{A}^\dagger$ works as raising operator and $\widehat{A}$ as lowering operator.

5 Super Partner Hamiltonians for Quaternion Harmonic Oscillator

Let us define the ground state wave function as

$$\psi_0^{(-)} = C\ \exp\left(\int \overrightarrow{U}(s).d\overrightarrow{s}\right) \tag{44}$$

where C is the normalization constant, $\overrightarrow{U}(s)$ is the quaternion potential and $d\overrightarrow{s}$ is quaternion difference operator so that

$$\overrightarrow{U}(s) = \omega(e_1x_1 + e_2x_2 + e_3x_3) \tag{45}$$

and ω is the frequency of the oscillator. Restricting the propagation in one dimension only, we get

$$\overrightarrow{U}(s) = \omega e_2x_2 = \omega e_2 x; \quad d\overrightarrow{s} = e_2 dx \tag{46}$$

Then (45) reduces to

$$\psi_0^{(-)} = C\ \exp\left(-\int \omega x.dx\right) = C\ \exp\left(-\frac{\omega x^2}{2}\right) \tag{47}$$

Super partner potentials are then be expressed as

$$\widehat{V}_-(x) = e_2\widehat{U}'(x) - \widehat{U}^2(x) = -\omega + \omega^2x^2,$$
$$\widehat{V}_-(x) = -e_2\widehat{U}'(x) - \widehat{U}^2(x) = \omega + \omega^2x^2 \tag{48}$$

and we get

$$\begin{aligned} \hat{H}_1 = \hat{H}_- &= -\frac{d^2}{dx^2} + \widehat{V}_-(x) = -\frac{d^2}{dx^2} - \omega + \omega^2x^2, \\ \hat{H}_2 = \hat{H}_+ &= -\frac{d^2}{dx^2} + \widehat{V}_+(x) = -\frac{d^2}{dx^2} + \omega + \omega^2x^2 \end{aligned} \tag{49}$$

It may readily be proved that $\widehat{U}^2(x)$ is proportional to anti commutation of annihilation operator $\widehat{A}$ and creation operator $\widehat{A}^\dagger$ while the first derivative $\widehat{U}'(x)$ is proportional to the commutation of annihilation operator $\widehat{A}$ and creation operator $\widehat{A}^{\dagger}$ multiplied by quaternion unit e_2. In the case of quaternionic quantum mechanics the Hamiltonians $\widehat{H}_+$ and $\hat{H}_-$ are super partner Hamiltonians i.e. for any eigen function $\psi_0^{(-)}$ of $\hat{H}_-$ with the corresponding eigenvalues E, $\widehat{A}\psi_0^{(-)}$ is an eigen function of $\widehat{H}_+$ with the same eigenvalue. Similarly, for any eigenfunction $\psi_0^{(+)}$ of $\widehat{H}_+$, $\widehat{A}^{\dagger}\psi_0^{(+)}$ is an eigenfunction of $\hat{H}_-$ with the same eigenvalue. We may now calculate the energy eigenvalue spectrum of Quaternion Harmonic Oscillator from the basic definition of supersymmetry. Using (47) and (48) we get

$$\hat{H}_1\psi_0^{(-)} = 0 = E_0^{(-)}\psi_0^{(-)} = E_0^{(1)}\psi_0^{(1)} \tag{50}$$

which shows that energy eigenvalue of $\hat{H}_-$ is zero. This eigenvalue can be considered as ground state energy and is the same as those obtained earlier[14] for the case of quaternion supersymmetric harmonic oscillator. Similarly, we may calculate the energy of super partner Hamiltonian $\widehat{H}_+$ or $\hat{H}_2$ as

$$\hat{H}_2\psi_0^{(-)} = 2\omega C \exp\left(-\frac{\omega x^2}{2}\right) = 2\omega\psi_0^{(-)} = E_0^{(2)}\psi_0^{(1)} \neq 0 \tag{51}$$

which shows that

$$E_0^{(2)} = E_0^{(+)} = 2\omega \tag{52}$$

It shows that ground state energy of $\widehat{H}_+(\hat{H}_{2)}$ is not zero. Accordingly we may calculate

$$\begin{aligned} E_0^{(+)} &= E_1^{(-)} = 2\omega, \\ E_2^{(-)} &= E_1^{(+)} = 4\omega \end{aligned} \tag{53}$$

and so on. In other words we may write the general relation between nth and $(n+1)$th energy levels in the following manner

$$E_{n+1}^{(-)} = E_n^{(+)} \tag{54}$$

Hence, the energy spectrum is related as the relation between two consecutive energy eigenstates which are equally spaced. Thus our results are same as those obtained earlier by Sukumar[10].

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